<u>Compactness theorem for propositional logic:</u>

A set of formulas $\mathcal{A}$ in a propositional language L is satisfiable if and only if it is finitely satisfiable.

<u>Corollary</u> Let L be a propositional language, $\mathcal{A}$ a set of formulas and B a formula. Then $\mathcal{A} \models B$, i.e., B is a tautological consequence of $\mathcal{A}$ if and only if there is a finite subset $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\mathcal{A}_0 \models B$.

<u>Proof</u> Note that $\mathcal{A} \models B$ if and only if $\mathcal{A} \cup \{\neg B\}$ is not satisfiable if and only if there is some finite set $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\mathcal{A}_0 \cup \{\neg B\}$ is not satisfiable. $\square$

A graph is an ordered pair $G = (V, E)$ where V are the <u>vertices</u> of G, $V \neq \emptyset$, and E are the <u>edges</u>, i.e., E is a collection of unordered pairs $\{x, y\}$, where $x \neq y$ are vertices.

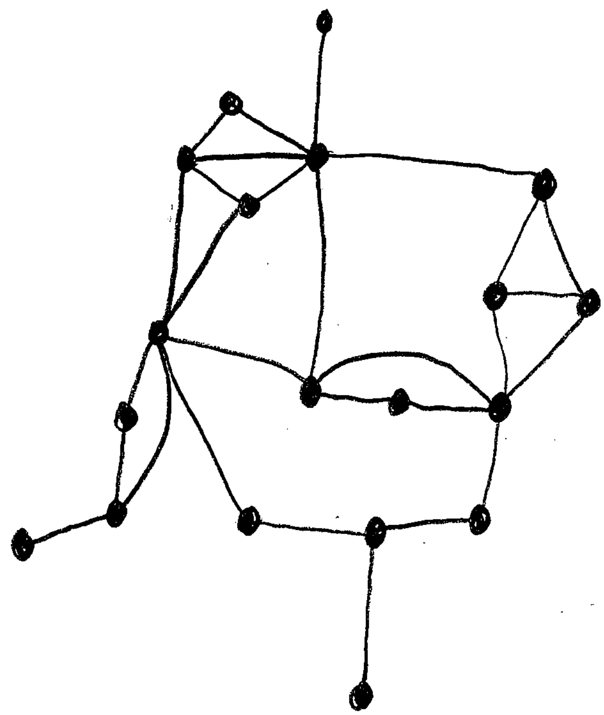
Given a subset $W \subseteq V$ of vertices, the <u>induced subgraph</u> $G|_W = (W, E')$ is simply given by $E' = \{ \{x,y\} \mid x, y \in W \}$.

<u>Example</u>

Consider a partial map of Europe and let $V = \{$countries on the map$\}$,

$\{x, y\} \in E \iff$ x and y are neighbours.

Thus the graph G looks as



A colouring of the graph $G = (V, E)$ with k colours is a function $c : V \longrightarrow \{1, 2, \dots, k\}$ such that if $\{x, y\} \in E$ then $c(x) \neq c(y)$.

Thus in the example above, this would correspond to colouring the map with k different colours such that neighbouring countries get different colours.

<u>Theorem</u> A graph $G = (V, E)$ admits a colouring with k colours if and only if every finite induced subgraph has a colouring with k colours.

<u>Proof</u> Let L be the language where variables are P_{xi} for every $x \in V$ and $i = 1, 2, \ldots, k$.

We let $\mathcal{A}$ be the set consisting of formulas

(1) $\quad P_{x1} \vee P_{x2} \vee \cdots \vee P_{xk}$ for $x \in V$

(2) $\quad \neg (P_{xi} \wedge P_{xj})$ for $x \in V$ and $i \neq j$

(3) $\quad \neg (P_{xi} \wedge P_{yi})$ for $i = 1, \ldots, k$ and $\{x, y\} \in E$.

Then if $v : L \to \{T, F\}$ is a valuation of L satisfying $\mathcal{A}$, we can define $c : V \to \{1, \ldots, k\}$ by setting

$$c(x) = i \iff v(P_{xi}) = T .$$

Since v satisfies $\mathcal{A}$, (i) implies that every vertex gets a colour, (ii) implies that a vertex gets only one colour and (iii) implies that vertices with an edge between them do not get the same colour.

So it is enough to show that $\mathcal{A}$ is satisfiable or, equivalently, that $\mathcal{A}$ is finitely satisfiable.

Suppose $\mathcal{A}_0 \subseteq \mathcal{A}$ is finite and let

$$W = \{x \in V \mid P_{xi} \text{ appears in some formula of } \mathcal{A}_0 \text{ for some } i=1,\dots,k\}.$$

Since $\mathcal{A}_0$ is finite, so is W and hence there is a colouring $c : W \longrightarrow \{1,2,\dots,k\}$ of the induced subgraph $G|_W$.

We let $v : L \longrightarrow \{T, F\}$ be defined by

$$v(P_{xi}) = T \iff x \in W \text{ and } c(x) = i$$
$$v(P_{xi}) = F \text{ otherwise.}$$

Then clearly v satisfies $\mathcal{A}_0$. $\square$

<u>Ramsey's theorem</u> (Infinite version)

For $A \subseteq \mathbb{N}$, set $[A]^2 = \{(n,m) \mid n,m \in A \text{ \& } n<m\}$.
We can think of $[A]^2$ as being the set of 2-element subsets of A.

<u>Thm</u> Suppose $c : [\mathbb{N}]^2 \longrightarrow \{\text{blue}, \text{red}\}$ is a colouring with two colours. Then there is an infinite subset $A \subseteq \mathbb{N}$ such that $[A]^2$ is monochromatic, i.e., $c|_{[A]^2}$ is constant.

<u>Exercise</u> Deduce Ramsey's Theorem from the above theorem using the compactness theorem for propositional logic!

<u>Ramsey's Theorem</u> For any k there is an n such that for any colouring
$$c : [\{1,2,\dots,n\}]^2 \longrightarrow \{\text{blue}, \text{red}\}$$
there is k-element subset $A \subseteq \{1,2,\dots,n\}$ such that $[A]^2$ is monochromatic.